

Interacting Bose and Fermi gases in low dimensions and the Riemann hypothesis

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Abstract

We apply the S-matrix based finite temperature formalism to non-relativistic Bose and Fermi gases in $1 + 1$ and $2 + 1$ dimensions. For the $2 + 1$ dimensional case in the constant scattering length approximation, the free energy is given in terms of Roger's dilogarithm in a way analagous to the thermodynamic Bethe ansatz for the relativistic $1 + 1$ dimensional case. The $1d$ fermionic case with a quasi-periodic 2-body potential could provide a physical framework for understanding the Riemann hypothesis.

I. INTRODUCTION

Quantum gases at finite temperature and density have very wide applications in nearly all areas of physics, ranging from black body radiation, Fermi liquids, Bose-Einstein condensation, and equations of state in cosmology. If one knows the zero temperature dynamics, i.e. the complete spectrum of the hamiltonian, then the quantum statistical mechanics just requires an additional statistical summation $Z = \text{Tr}(e^{-\beta H})$, so that in principle zero temperature dynamics and quantum statistical sums are decoupled. This decoupling is also in principle clear from an intuitive picture of a gas as a finite density of particles that are subject to scattering. In practice, the complete non-perturbative spectrum of H is unknown so one must resort to perturbative methods, such as the Matsubara approach, that typically entangle the zero temperature dynamics from the quantum statistical mechanics.

In [1] an alternative approach to finite temperature quantum field theory was developed that achieves this decoupling of zero temperature dynamics and quantum statistical summation. The dynamical variables are the occupation numbers, or filling fractions, of the gas, and the zero temperature data of the underlying theory are just the S-matrix scattering amplitudes. Our construction was modeled after Yang and Yang's thermodynamic Bethe ansatz (TBA)[2]. The TBA is very specific to integrable theories in one spacial dimension. Nevertheless, our derivation [1] is carried out in any spacial dimension d and doesn't assume integrability; therefore it is in general an approximate method, though it can be systematically improved. An important ingredient of our construction is the work of Dashen, Ma, and Bernstein[3] which explains how to formulate quantum statistical mechanics in terms of the S-matrix. Ideas of Lee and Yang were also instrumental[4], even though the latter work is not based on the S-matrix.

Our previous work[1] was mainly devoted to developing the formalism in a general, model independent way for both non-relativistic and relativistic theories. Though the derivation was somewhat involved, the final result is straightforward to implement and is summarized by the two formulas (23, 24). In this paper we apply the method to non-relativistic Bose and Fermi gases with special attention paid to the low dimensional cases of $d = 1, 2$. The main approximation we make is to consider only the contributions to the free energy that come from 2-particle to 2-particle scattering. On physical grounds these are expected to be the most important if the gas is not too dense. Even in this approximation the problem

is non-trivial since one needs to resum all the 2-body interactions self-consistently. This is accomplished by an integral equation for the filling fractions that is analogous to the TBA equations.

In the next section we first review the main results in [1]. In section III we describe the kind of second-quantized hamiltonians that are the subject of this paper. In section IV we consider bosons with a δ function two-body potential. In section V we turn to fermionic gases in the same approximation and derive formulas that can be used to study the effect of interactions on the Fermi energy. Since the lowest order contribution corresponds to a constant scattering length approximation, the results we obtain in sections IV and V are already known, and are included here mainly for illustrative purposes.

In section IV we study the $2d$ case which has some remarkable features. The integral equation which determines the filling fraction becomes algebraic. The formulas for the free energy are essentially identical to those for the conformally invariant limit of *relativistic* theories in *one lower* dimension $d = 1$ [5] and are given in terms of Roger's dilogarithm. In making this analogy we define a "central charge" c as the coefficient in the free energy, which in the relativistic case in $1d$ is the Virasoro central charge. As in the $1d$ relativistic case, for certain special values of the coupling, c can be a rational number, and some examples based on the golden mean are presented. We also extend the formalism to many species of mixed bosonic and fermionic statistics.

As will be clear in the sequel, the polylogarithmic and Riemann ζ functions play a central role in this work. This provided us with a new opportunity to understand Riemann's hypothesis[6]. Riemann's ζ function for $Re(\nu) > 1$ is defined as

$$\zeta(\nu) = \sum_{n=1}^{\infty} \frac{1}{n^{\nu}} \quad (1)$$

It has a simple pole at $\nu = 1$; as we will describe this pole is a manifestation of the impossibility of Bose-Einstein condensation in $2d$. The function can be analytically continued to the whole complex ν plane. It has trivial zeros at $\nu = -2n$ where n is a non-zero positive integer. The function has a duality relation that relates $\zeta(\nu)$ to $\zeta(1 - \nu)$ (eq. (109) below) so that its non-trivial zeros are in the critical strip $0 < Re(\nu) < 1$ and symmetric around $Re(\nu) = 1/2$. The Riemann hypothesis (RH) is the statement that the only non-trivial zeros of $\zeta(\nu)$ are at $Re(\nu) = 1/2$. This hypothesis is very important in number theory since, as shown by Riemann, the distribution of prime numbers is intimately related to the location

of the Riemann zeros. It has already been proven by Hadamard and de la Vallée-Poussin that there are no zeros with $Re(\nu) = 1$, which is equivalent to proving the Prime Number Theorem[6].

There have been a number of approaches to understanding the Riemann hypothesis based on physics[26]. Here we mention a few of them mainly to contrast with the approach presented here. Some approaches are inspired by the Polya-Hilbert conjecture which supposes that the imaginary part α of the zeros $\nu = 1/2 + i\alpha$ corresponds to eigenenergies of some unknown quantum mechanical hamiltonian. The well-studied statistical properties of the zeros, in particular the Montgomery-Odlyzko law and its relation to the gaussian unitary ensemble of random matrices[7, 8], led Berry to propose a quantum chaos interpretation of the oscillatory part of Riemann's counting formula for the zeros[9]. This is still within the context of quantum mechanics at zero temperature. Connes has proposed the complementary picture that the zeros correspond to absorption lines[10]. Sierra has recently proposed a consistent quantization of the Berry-Keating hamiltonian based on the Russian doll model of superconductivity, which possesses a cyclic renormalization group flow[11].

The fundamental duality relation relating $\zeta(\nu)$ to $\zeta(1 - \nu)$ can be understood as a consequence of a special modular transformation for the quantum statistical mechanics of free, relativistic, massless particles in $\nu = d + 1$ spacetime dimensions. This is explained in the appendix, since it is tangential to the main development of this paper. In section VII Riemann's ζ function on the critical strip is related to the quantum statistical mechanics of *non-relativistic*, interacting *fermionic* gases in $1d$ with a quasi-periodic 2-body potential which depends on ν . This is thus a many-body problem at finite temperature. It is perhaps not completely unanticipated that the RH could have a resolution in the present context, since one of the very first places in physics where ζ appeared was in Planck's work on black body radiation, which is widely acknowledged as the birth of quantum mechanics[12]. The black body theory is the bosonic quantum statistical mechanics of a $3d$ gas of free relativistic photons[27]. In this case what appears is $\zeta(4) = \pi^4/90$. As we describe here, in order to get into the critical strip one needs $d = 1$ non-relativistic fermions interacting with a quasi-periodic potential. The fermionic nature renders the required integrals convergent and is also needed for a well-defined hamiltonian. This quasi-periodicity implies our approach is closest to Sierra's[11], since there also a periodicity was important, but in the renormalization group at zero temperature. Our approach is thus essentially different just because

the physical context is different, however our interest in the RH actually stemmed from the work[13] on cyclic renormalization group flows in $1d$ relativistic systems and the observation that the finite temperature behavior was in part characterized by $\zeta(1 - i\alpha)$, where $2\pi/\alpha$ is the period of the renormalization group flow[14]. We now understand that one needs to consider non-relativistic models to get $Re(\nu) < 1$ in a natural way[28].

The understanding of the RH that emerges from the present context can be summarized as follows. When $Re(\nu) > 1/2$, there exists a two-body potential in position space that leads to a well-defined quasi-periodic kernel \mathbf{K} in momentum space that is essentially a combination of Fourier transforms of the potential. The condition $Re(\nu) > 1/2$ comes about naturally as the condition for the convergence of the appropriate Fourier transform. The quantum statistical mechanics based on this kernel gives corrections to the pressure of the gas that are determined by a transcendental equation involving the polylogarithm Li_ν . To obtain this, one must work with the momentum-dependent kernel, i.e. the constant scattering length approximation made in earlier sections vanishes. If $\zeta(\nu) = 0$ then there would exist solutions with vanishing corrections to the pressure. Therefore the RH would follow from the simple physical property that non-zero interactions necessarily modify the pressure. We comment on the meaning of the actual zeros in section VII.

II. FREE ENERGY AS A DYNAMICAL FUNCTIONAL OF FILLING FRACTION

A. Generalities

The free energy density (per volume) \mathcal{F} is defined as

$$\mathcal{F} = -\frac{1}{\beta V} \log Z, \quad Z = \text{Tr } e^{-\beta(H - \mu N)} \quad (2)$$

where $\beta = 1/T$, μ are the inverse temperature and chemical potential, V is the d -dimensional spacial volume, and H and N are the hamiltonian and particle number operator. Since $\log Z$ is an extensive quantity, i.e. proportional to the volume, the pressure p of the gas is minus the free energy density:

$$p = T \frac{d \log Z}{dV} = -\mathcal{F} \quad (3)$$

For most of this paper, we assume there is one species of bosonic ($s = 1$) or fermionic ($s = -1$) particle. Given $\mathcal{F}(\mu)$, one can compute the thermally averaged number density n :

$$n = -\frac{\partial \mathcal{F}}{\partial \mu} \equiv \int \frac{d^d \mathbf{k}}{(2\pi)^d} f(\mathbf{k}) \quad (4)$$

where \mathbf{k} is the d -dimensional momentum. The dimensionless quantities f are called the filling fractions or occupation numbers.

One can express \mathcal{F} as a functional of f in a meaningful way with a Legendre transformation. Define

$$G \equiv \mathcal{F}(\mu) + \mu n \quad (5)$$

Treating f and μ as independent variables, then using eq. (4) one has that $\partial_\mu G = 0$ which implies it can be expressed only in terms of f and satisfies $\delta G / \delta f = \mu$.

Inverting the above construction shows that there exists a functional $F(f, \mu)$

$$F(f, \mu) = G(f) - \mu \int \frac{d^d \mathbf{k}}{(2\pi)^d} f(\mathbf{k}) \quad (6)$$

which satisfies eq. (4) and is a stationary point with respect to f :

$$\frac{\delta F}{\delta f} = 0 \quad (7)$$

The above stationary condition is to be viewed as determining f as a function of μ . The physical free energy is then $\mathcal{F} = F$ evaluated at the solution f to the above equation. We will refer to eq. (7) as the saddle point equation since it is suggestive of a saddle point approximation to a functional integral:

$$Z = \int Df e^{-\beta V F(f)} \approx e^{-\beta V \mathcal{F}} \quad (8)$$

In a free theory, the eigenstates of H are multi-particle Fock space states $|\mathbf{k}_1, \mathbf{k}_2, \dots\rangle$. Let $\omega_{\mathbf{k}}$ denote the one-particle energy as a function of momentum \mathbf{k} . In this paper the theory is assumed to be non-relativistic with

$$\omega_{\mathbf{k}} = \frac{\mathbf{k}^2}{2m} \quad (9)$$

where m is the mass of the particle. It is well-known that the trace over the multi-particle Fock space gives

$$\mathcal{F}_0(\mu) = \frac{s}{\beta} \int \frac{d^d \mathbf{k}}{(2\pi)^d} \log(1 - s e^{-\beta(\omega_{\mathbf{k}} - \mu)}) \quad (10)$$

From the definition eq. (4) one finds the filling fractions:

$$f(\mathbf{k}) = \frac{1}{e^{\beta(\omega_{\mathbf{k}} - \mu)} - s} \equiv f_0(\mathbf{k}) \quad (11)$$

In order to find the functional $F(f, \mu)$ one first computes G from eq. (5) and eliminates μ to express it in terms of f using eq. (11). One finds

$$F_0(f, \mu) = \int \frac{d^d \mathbf{k}}{(2\pi)^d} \left((\omega_{\mathbf{k}} - \mu)f - \frac{1}{\beta} \left[(f + s) \log(1 + sf) - f \log f \right] \right) \quad (12)$$

One can then easily verify that $\delta F / \delta f = 0$ has the solution $f = f_0$ and plugging this back into eq. (12) gives the correct result eq. (10) for \mathcal{F}_0 . In the sequel, it will be convenient to trade the chemical potential variable μ for the variable f_0 :

$$F_0(f, f_0) = -\frac{1}{\beta} \int \frac{d^d \mathbf{k}}{(2\pi)^d} \left(s \log(1 + sf) + f \log \left(\frac{1 + sf}{f} \frac{f_0}{1 + sf_0} \right) \right) \quad (13)$$

There is another way to view the above construction which involves the entropy. Write eq. (12) as

$$F = \mathcal{E} - \frac{1}{\beta} \mathcal{S} \quad (14)$$

where \mathcal{E} is the first $(\omega - \mu)f$ term in eq. (12), which is the energy density, and \mathcal{S} is the remaining term in brackets. One can show by a standard counting argument, which involves the statistics of the particles, that \mathcal{S} represents the entropy density of a gas of particles. (See for instance [15].)

Let us now include interactions by writing

$$F(f, f_0) = F_0(f, f_0) + F_1(f) \quad (15)$$

where F_0 is given in eq. (13) and we define U as the “potential” which depends on f and incorporates interactions:

$$F_1 = -\frac{1}{\beta} \int \frac{d^d \mathbf{k}}{(2\pi)^d} U(f(\mathbf{k})) \quad (16)$$

Given F , f is determined by the saddle point equation $\delta F / \delta f = 0$. It is convenient to define a pseudo-energy ε as the following parameterization of f :

$$f \equiv \frac{1}{e^{\beta\varepsilon} - s} \quad (17)$$

Then the saddle point equation and free energy density take the form:

$$\varepsilon = \omega - \mu - \frac{1}{\beta} \frac{\partial U}{\partial f} \quad (18)$$

$$\mathcal{F} = -\frac{1}{\beta} \int \frac{d^d \mathbf{k}}{(2\pi)^d} \left[-s \log(1 - se^{-\beta\varepsilon}) + (1 - f \partial_f) U \right] \quad (19)$$

B. Two-body approximation

It was shown in [1] how to express U in terms of S-matrix scattering amplitudes. In general there are terms involving the forward scattering of $n \rightarrow n$ numbers of particles for $n \geq 2$. The most important is the two-body contribution. It takes the form

$$U(\mathbf{k}) = \frac{\beta}{2} f(\mathbf{k}) (\mathbf{K} * f)(\mathbf{k}) \quad (20)$$

where we have defined the convolution

$$(\mathbf{K} * f)(\mathbf{k}) \equiv \int \frac{d^d \mathbf{k}'}{(2\pi)^d} \mathbf{K}(\mathbf{k}, \mathbf{k}') f(\mathbf{k}'), \quad (21)$$

and the kernel \mathbf{K} is given by the 2-particle to 2-particle forward scattering amplitude:

$$\mathbf{K}(\mathbf{k}, \mathbf{k}') \equiv \mathcal{M}_{12;12}(\mathbf{k}, \mathbf{k}') \quad (22)$$

How to compute the kernel from the hamiltonian is described below.

In terms of the pseudo-energy, the saddle point equation and free energy take the following forms:

$$\varepsilon = \omega - \mu - \mathbf{K} * \left(\frac{1}{e^{\beta\varepsilon} - s} \right) \quad (23)$$

$$\mathcal{F} = -\frac{1}{\beta} \int \frac{d^d \mathbf{k}}{(2\pi)^d} \left[-s \log(1 - s e^{-\beta\varepsilon}) + \frac{\beta f}{2} (\varepsilon - \omega + \mu) \right] \quad (24)$$

III. SECOND QUANTIZED HAMILTONIANS

In this paper we consider second quantized hamiltonians of the form:

$$H = \int d^d \mathbf{x} \left(\frac{1}{2m} \vec{\nabla} \psi^\dagger \cdot \vec{\nabla} \psi \right) + \frac{1}{4} \int d^d \mathbf{x} \int d^d \mathbf{x}' \mathcal{V}(\mathbf{x} - \mathbf{x}') \psi^\dagger(\mathbf{x}) \psi^\dagger(\mathbf{x}') \psi(\mathbf{x}') \psi(\mathbf{x}) \quad (25)$$

where \mathcal{V} is the 2-body potential. The field satisfies the canonical commutation relations

$$\psi(\mathbf{x}) \psi^\dagger(\mathbf{x}') - s \psi^\dagger(\mathbf{x}') \psi(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}') \quad (26)$$

where again $s = \pm 1$ corresponds to bosons/fermions. Expanding the field in terms of annihilation operators,

$$\psi(\mathbf{x}) = \int \frac{d^d \mathbf{k}}{(2\pi)^d} e^{i\mathbf{k} \cdot \mathbf{x}} a_{\mathbf{k}} \quad (27)$$

this leads to the canonical (anti-) commutation relations:

$$a_{\mathbf{k}} a_{\mathbf{k}'}^\dagger - s a_{\mathbf{k}'}^\dagger a_{\mathbf{k}} = (2\pi)^d \delta(\mathbf{k} - \mathbf{k}') \quad (28)$$

The free Hilbert space is thus a bosonic or fermionic Fock space with momentum eigenstates normalized as: $|\mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_n\rangle = a_{\mathbf{k}_1}^\dagger \cdots a_{\mathbf{k}_n}^\dagger |0\rangle$.

In this paper we only consider the lowest order contribution to the kernel \mathbf{K} . Let H_1 denote the interacting part of H which depends on the 2-body potential \mathcal{V} . Then to lowest order,

$$\langle \alpha' | H_1 | \alpha \rangle = -(2\pi)^d \delta^{(d)}(\mathbf{k}_\alpha - \mathbf{k}_{\alpha'}) \mathcal{M}_{\alpha'; \alpha} \quad (29)$$

where \mathcal{M} is the scattering amplitude for the asymptotic states $|\alpha\rangle$. Therefore to lowest order the kernel is given by

$$\mathbf{K}(\mathbf{k}, \mathbf{k}') = -V^{-1} \langle \mathbf{k}, \mathbf{k}' | H_1 | \mathbf{k}, \mathbf{k}' \rangle \quad (30)$$

IV. HARD CORE BOSON MODEL REVISITED

In this section we consider bosonic particles with a delta-function two-body potential:

$$\mathcal{V}(\mathbf{x}, \mathbf{x}') = \gamma \delta^{(d)}(\mathbf{x} - \mathbf{x}') \quad (31)$$

The model has been solved exactly in $1d$ by Lieb and Liniger[16], and the thermodynamic Bethe ansatz was first discovered in the context of this model[2].

Using $(2\pi)^d \delta^{(d)}(0) = V$, one finds

$$\mathbf{K} = -\gamma \quad (32)$$

Positive γ corresponds to repulsive interactions. Since the kernel is constant in this approximation, it is equivalent to the constant scattering length approximation made in the literature; hence we do not obtain any new results for the hard core Bose gas here. We include this section and the next mainly for illustrative purposes and as a warm-up to the sequel.

The coupling constant γ has units of energy \times volume. It can be expressed in terms of a physical scattering length a as follows. To first order in perturbation theory the differential cross-section in the center of mass is:

$$\frac{d\sigma}{d\Omega} = \frac{m^2 \gamma^2}{4(2\pi)^{d-1}} k^{d-3} \quad (33)$$

where k is the magnitude of \mathbf{k} for one of the incoming particles. Since a cross section has dimensions of length ^{$d-1$} , we define a such that the cross section is a^{d-1} when the wavelength of the particle is $2\pi/a$:

$$\left. \frac{d\sigma}{d\Omega} \right|_{k \sim 2\pi/a} \sim a^{d-1} \quad (34)$$

This leads us to make the definition:

$$\frac{\gamma}{(2\pi)^{d/2}} \equiv \frac{a^{d-2}}{m} \quad (35)$$

We carry out our analysis for arbitrary spacial dimension $d > 0$. Using $\int d^d \mathbf{k} = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int dk k^{d-1}$ where Γ is the standard Γ -function, rotationally invariant integrals over momenta \mathbf{k} can be traded for integrals over $\omega_{\mathbf{k}}$:

$$\int \frac{d^d \mathbf{k}}{(2\pi)^d} = \left(\frac{m}{2\pi} \right)^{d/2} \frac{1}{\Gamma(d/2)} \int_0^\infty d\omega \omega^{(d-2)/2} \quad (36)$$

For a constant kernel $\mathbf{K} = -\gamma$, and $d > 0$, the solution to the integral equation eq. (23) takes the simple form:

$$\varepsilon(\mathbf{k}) = \omega_{\mathbf{k}} - \mu + T\delta \quad (37)$$

where δ is independent of \mathbf{k} . To determine δ one needs the integral:

$$\int_0^\infty dx \frac{z x^{\nu-1}}{e^x - z} = \Gamma(\nu) \text{Li}_\nu(z) \quad \text{for } \text{Re}(\nu) > 0 \quad (38)$$

The function $\text{Li}_\nu(z)$ is the standard polylogarithm, defined as the appropriate analytic continuation of

$$\text{Li}_\nu(z) = \sum_{n=1}^\infty \frac{z^n}{n^\nu} \quad (39)$$

For fixed ν , the function $\text{Li}_\nu(z)$ has a branch point at $z = 1$ with a cut along $\text{Re}(z) > 1$, and the above integral is not valid at $z = 1$. We will later need the fact that when $z = 1$, the above integral is only convergent when $\text{Re}(\nu) > 1$ and is given in terms of Riemann's ζ function:

$$\int_0^\infty dx \frac{x^{\nu-1}}{e^x - 1} = \Gamma(\nu) \zeta(\nu), \quad \text{Re}(\nu) > 1 \quad (40)$$

Equivalently,

$$\text{Li}_\nu(1) = \zeta(\nu), \quad \text{Re}(\nu) > 1 \quad (41)$$

Eq. (23) then leads to the following equation satisfied by δ :

$$\delta = h_T \text{Li}_{d/2}(z_\mu z_\delta) \quad (42)$$

where we have defined the fugacities

$$z_\mu \equiv e^{\beta\mu}, \quad z_\delta = e^{-\delta} \quad (43)$$

and a renormalized thermal coupling h_T and thermal wavelength λ_T :

$$h_T \equiv \left(\frac{\sqrt{2\pi}a}{\lambda_T} \right)^{d-2}, \quad \lambda_T \equiv \sqrt{\frac{2\pi}{mT}} \quad (44)$$

To simplify subsequent expressions, we henceforth set the mass $m = 1/2$, unless otherwise stated. This leads us to define:

$$\tilde{T} \equiv \frac{1}{\lambda_T^2} = \frac{T}{4\pi}, \quad (m = 1/2) \quad (45)$$

The equation (42) is a transcendental equation that determines δ as a function of μ, T , and the coupling h_T . Given the solution $\delta(\mu)$ of this equation, using eq. (24) the density can be expressed as

$$n(\mu) = \tilde{T}^{d/2} \text{Li}_{d/2}(z_\mu z_\delta) \quad (46)$$

For $d > 0$ one can integrate by parts and obtain the following expressions for the free energy

$$\mathcal{F} = -T \tilde{T}^{d/2} \left(\text{Li}_{(d+2)/2}(z_\mu z_\delta) + \frac{\delta}{2} \text{Li}_{d/2}(z_\mu z_\delta) \right) \quad (47)$$

Bose-Einstein condensation can be described rather generally as follows. The property that an extensive, i.e. proportional to the volume, number of particles occupy the ground state with $|\mathbf{k}| = 0$ implies that the filling fraction f diverges at some critical chemical potential μ_c :

$$\lim_{\mu \rightarrow \mu_c} f(|\mathbf{k}| = 0, \mu) = \infty \quad (48)$$

Given μ_c , one can speak of a critical density n_c :

$$n_c(T) = \int \frac{d^d \mathbf{k}}{(2\pi)^d} f(\mathbf{k}, \mu_c) \quad (49)$$

One also has a critical temperature T_c defined as

$$n_c(T_c) = n_{\text{phys.}} \quad (50)$$

where $n_{\text{phys.}}$ is the physical density. The reason that T_c corresponds to a transition is that because of the divergence, one must treat the density of particles in the ground state n_{gs} separately from the above expressions for n :

$$n_{\text{phys.}} = n_{gs} + n \quad (51)$$

where n_{gs} is the density of particles in the ground state and n is the integral of f . When $T > T_c$, n_{gs} vanishes. The number of particles in the ground state doesn't actually vanish, but rather it is no longer proportional to the volume, so that n_{gs} effectively vanishes.

In the pseudo-energy description (17) of the filling fraction f , the condition (48) is just

$$\varepsilon(|\mathbf{k}| = 0, \mu = \mu_c) = 0 \quad (52)$$

Eq. (23) then leads to the equation for μ_c : $\mu_c = T\delta(\mu_c)$. In terms of the fugacities the critical point is the simple relation

$$z_{\mu_c} z_{\delta(\mu_c)} = 1 \quad (53)$$

At the critical point, the arguments of the polylogarithms are 1, and if $d > 2$, then $\text{Li}_{d/2}(1) = \zeta(d/2)$. (Eq. (41)). This in turn implies that the critical temperature and density are independent of the coupling h_T :

$$\mu_c = h_T \zeta(d/2) \quad (54)$$

$$n_c = \zeta(d/2) \tilde{T}^{d/2} \quad (55)$$

$$T_c = 4\pi \left(\frac{n_{\text{phys.}}}{\zeta(d/2)} \right)^{2/d} \quad (56)$$

Though the critical temperature and density do not depend on the coupling h_T to this order, the free energy at the critical point has corrections that do:

$$\mathcal{F} = -\zeta\left(\frac{d+2}{2}\right) T \tilde{T}^{d/2} \left(1 + \frac{h_T}{2} \frac{\zeta\left(\frac{d}{2}\right)^2}{\zeta\left(\frac{d+2}{2}\right)} \right) \quad (57)$$

The $\zeta(\nu)$ function has a simple pole at $\nu = 1$, and this has a physical significance in the present context. In particular, it implies that the critical density n_c is infinite and the critical temperature T_c is zero in $2d$ to the order we have calculated. This means that in $3d$ Bose particles have a stronger tendency to Bose-Einstein condense in comparison with $2d$. This simple pole at $\nu = 1$ can thus be interpreted as a manifestation of the Mermin-Wagner theorem, which states that finite temperature continuous phase transitions are not possible in $2d$ [18]. Another way of viewing this is that in $2d$, bosonic particles behave more like fermions and don't readily Bose-Einstein condense. This fact is ultimately responsible for why one needs to treat bosonic particles with fermionic exclusion statistics in $1d$ as far as their quantum statistical mechanics is concerned. See for instance [17]. For relativistic models also, it appears that only the fermionic thermodynamic Bethe ansatz equations are consistent[19, 20].

V. FERMI GASES AND THE FERMI ENERGY

In this section we consider fermionic particles with the same constant kernel $\mathbf{K} = -\gamma$ as in the last section. To obtain such a kernel from the second quantized hamiltonian one needs at least two species of fermions since $\psi^2 = 0$, however for simplicity we consider only one species.

The integral we need is now:

$$\int_0^\infty dx \frac{z x^{\nu-1}}{e^x + z} = -\Gamma(\nu) \text{Li}_\nu(-z) \quad \text{Re}(\nu) > 0 \quad (58)$$

It is important to realize that contrary to the bosonic case, eq. (40), the above integral is valid when $z = 1$ throughout the critical strip $0 < \text{Re}(\nu) < 1$ and is given by the above formula where:

$$-\text{Li}_\nu(-1) = (1 - 2^{1-\nu})\zeta(\nu) \quad \forall \nu \quad (59)$$

In physical terms, fermions are much more stable than bosons in $d < 2$ because of the Pauli exclusion principle.

The pseudo-energy still takes the form (37), and the saddle point equation (23) now leads to the equation:

$$\delta = -h_T \text{Li}_{d/2}(-z_\mu z_\delta) \quad (60)$$

The density and free energy can be expressed as:

$$\begin{aligned} n &= -\tilde{T}^{d/2} \text{Li}_{d/2}(-z_\mu z_\delta) \\ \mathcal{F} &= T \tilde{T}^{d/2} \left(\text{Li}_{(d+2)/2}(-z_\mu z_\delta) + \frac{\delta}{2} \text{Li}_{d/2}(-z_\mu z_\delta) \right) \end{aligned} \quad (61)$$

In the fermionic case, the considerations of the last section are replaced by the concept of a Fermi surface. As we'll see, mathematically there are analogies between the existence of the Fermi surface and Bose-Einstein condensation. The Fermi energy ω_F is defined as the uppermost energy that is occupied, i.e. has $f \neq 0$. In the limit of zero temperature and no interactions, f is a step function: $f(\mathbf{k}) = 1$ for $|\mathbf{k}| < |\mathbf{k}_F|$ and zero otherwise, where $\omega_F = \mathbf{k}_F^2/2m$. At finite temperature the sharp step is broadened.

The formulas eqs. (60,61) can be used to study the effect of interactions and finite temperature on ω_F . The Fermi energy naturally can be defined as the point where $f = 1/2$, i.e. where $\varepsilon = 0$:

$$\varepsilon(\mathbf{k}_F) = 0 = \omega_F - \mu + T\delta \quad (62)$$

The latter implies:

$$z_\mu z_\delta = e^{\beta\omega_F} \quad (63)$$

The Fermi energy may now be expressed in terms of the density n :

$$n = -\tilde{T}^{d/2} \text{Li}_{d/2}(-e^{\beta\omega_F}) \quad (64)$$

The above formula determines ω_F as a function of T and n . As the temperature goes to zero, one can use

$$\lim_{T \rightarrow 0} \text{Li}_\nu(-e^{\omega_F/T}) \approx -\frac{(\omega_F/T)^\nu}{\Gamma(\nu+1)} \quad (65)$$

to obtain

$$\omega_F = 4\pi \left(\Gamma\left(\frac{d+2}{2}\right) n \right)^{2/d} \quad (66)$$

In $3d$ the above formula is equivalent to well-known results[15].

Note that, to the order we have done the computation, the Fermi energy does not depend on the interactions since eq. (64) no longer depends on the interaction. However in the presence of interactions $\delta \neq 0$, the free energy is modified from its non-interacting value, as eq. (61) shows.

VI. $2d$ BOSE AND FERMI GASES

In this section we consider the $2d$ case which is rather special. Referring to eq. (44) the coupling h_T is temperature independent in $2d$, and is in fact dimensionless. This implies that in $2d$ the scattering length a^{d-2} should simply be replaced by a dimensionless coupling we will denote as h . Many of the previous formulas are especially simple due to the identity $\text{Li}_1(z) = -\log(1-z)$. We also describe a strong similarity to the conformally invariant limit of relativistic systems in one lower dimension.

A. Bosonic case

The equation (42) for δ becomes algebraic:

$$z_\delta = (1 - z_\mu z_\delta)^h \quad (67)$$

The density is simply a logarithm

$$n = -\frac{T}{4\pi} \log(1 - z_\mu z_\delta) \quad (68)$$

and the free energy is expressed in terms of the dilogarithm:

$$\mathcal{F} = -\frac{T^2}{4\pi} \left(\text{Li}_2(z_\mu z_\delta) - \frac{\delta}{2} \log(1 - z_\mu z_\delta) \right) \quad (69)$$

The density is well defined and positive at zero chemical potential μ , so long as $0 < z_\delta < 1$. The equation (67) has a real solution in this range as long as $h > 0$, i.e. as long as the gas is repulsive. This observation means that the bosonic instability described in section IV can be cured by repulsive interactions. For the remainder of this section we set $z_\mu = 1$ and denote z_δ simply as z . Let us define c as the following coefficient in the free energy:

$$\mathcal{F} = -\frac{c\pi T^2}{24} \quad (70)$$

This is analogous to the relativistic case in $1d$, where one defines $\mathcal{F} = -c\pi T^2/6$ where in that context c is the Virasoro central charge[23, 24]. In terms of the solution to the equation (67), c is given in terms of Roger's dilogarithm:

$$c(z) = \frac{6}{\pi^2} \text{Lr}_2(z) \quad (71)$$

where

$$\text{Lr}_2(z) = \text{Li}_2(z) + \frac{1}{2} \log|z| \log(1 - z) \quad (72)$$

Interestingly, the above formula for c in terms of Lr_2 is identical to the formulas that arise when one studies the conformal limit of relativistic TBA systems in $1d$ [5, 19].

The function Lr_2 is known to satisfy the following functional relations[25]:

$$\text{Lr}_2(z) + \text{Lr}_2(1 - z) = \zeta(2) = \frac{\pi^2}{6} \quad (73)$$

$$\text{Lr}_2(z) + \text{Lr}_2(-z/(1 - z)) = 0$$

Using the first relation, one sees that $\text{Lr}_2(1/2) = \zeta(2)/2$. This, and the free fermion and free boson cases ($h = 0$) were known to Euler[21]. Landen found more relations as follows[22]. If r is a root to the polynomial equation $z^2 + z = 1$, then the above functional equations become linear equations for $\text{Lr}_2(z)$ with argument $z = r, -r, -1/r$ and r^2 . Let us chose $r = (\sqrt{5} - 1)/2$ which is the golden mean. When z is positive, the above relations imply that for special values of the coupling h , c is a rational number:

$$\begin{aligned} h &= 0, & z &= 1, & c &= 1 \\ h &= 1/2, & z &= r, & c &= 3/5 \\ h &= 1, & z &= 1/2, & c &= 1/2 \\ h &= 2, & z &= r^2, & c &= 2/5 \end{aligned} \quad (74)$$

The above relations may be viewed as results in additive number theory by virtue of eq. (39).

Since the pressure $p = -\mathcal{F}$, the coefficient c is a measure of the pressure of the gas. One observes from eq. (74) that as h increases the pressure decreases. This is expected on physical grounds: larger h means stronger repulsive interactions so the gas is less dense.

B. Fermionic case

In this case the equation that determines z (at zero chemical potential) is

$$z = (1 + z)^{-h} \quad (75)$$

The density takes the form

$$n = \frac{T}{4\pi} \log(1 + z) \quad (76)$$

The free energy has the same form as in eq. (70) where now

$$c = -\frac{6}{\pi^2} \text{Lr}_2(-z) \quad (77)$$

Because of the Pauli exclusion principle for fermions, the fermionic gas is more stable than the bosonic one. First note that unlike the bosonic case, the eq. (75) continues to have real solutions even if the interactions of the gas are attractive with $h < 0$. Another important feature is that whereas in the bosonic case $\text{Lr}_2(z)$ has a branch cut along the real z axis from 1 to ∞ , there are no branch points for z along the negative axis. So for fermions, the free energy is well defined for any real $z < 1$.

The special rational points of the bosonic case also have a fermionic version:

$$\begin{aligned} h &= -1, & z &= \infty, & c &= 1 \\ h &= -1/2, & z &= 1/r, & c &= 3/5 \\ h &= 0, & z &= 1, & c &= 1/2 \\ h &= 1, & z &= r, & c &= 2/5 \end{aligned} \quad (78)$$

Note here also that increasing h decreases the pressure since the gas is either less attractive or more repulsive.

C. Many mixed particles

For far we have only considered a single bosonic or fermionic particle. It is straightforward to extend this to many types of particles of mixed statistics. Let m_a and $s_a = \pm 1$ denote the mass and statistics of the a -th particle. In the two-body approximation, we consider the following contribution to F_1 :

$$U = \frac{\beta}{2} \sum_{a,b} f_a(\mathbf{k}) (\mathbf{K}_{ab} * f_b)(\mathbf{k}) \quad (79)$$

where \mathbf{K}_{ab} is the scattering amplitude of a with b types of particles. The saddle point equation now reads:

$$\varepsilon_a = \omega_a - \mu_a - \sum_b \mathbf{K}_{ab} * \frac{1}{e^{\beta \varepsilon_b} - s_b} \quad (80)$$

For a constant kernel \mathbf{K}_{ab} the above equation has the solution:

$$\varepsilon_a = \omega_a - \mu_a + T \delta_a \quad (81)$$

In $2d$ the equation satisfied by δ_a is again algebraic:

$$z_a = \prod_b (1 - s_b z_b)^{h_{ab} s_b} \quad (82)$$

where $z_a = e^{-\delta_a}$ and h_{ab} are dimensionless coupling constants that parameterize the kernel \mathbf{K}_{ab} . The density n_a of the a -th species is given by

$$n_a = -s_a \frac{m_a T}{2\pi} \log(1 - s_a z_a) \quad (83)$$

The total free energy is given by eq. (70) with

$$c = 2 \sum_a m_a c_{s_a}(z_a) \quad (84)$$

where c_+ is the bosonic expression (71) and c_- the fermionic one (77).

There exists a vast number of examples where certain choices of m_a , s_a , and h_{ab} lead to rational c . For instance, one can translate known fermionic relativistic systems in $1d$ to the present $2d$ non-relativistic context. Some of the latter are known to be related to root systems of Lie algebras. (See for instance [19].) There are also many more examples that follow from results in [25], not all of which have $1d$ relativistic analog. For purposes of illustration, consider a two-particle theory with one boson and one fermion, of equal mass

$1/2$, and with $h_{ab} = 1, \forall a, b$. This structure suggests a supersymmetric theory. The z_1, z_2 are solutions of

$$z_1 = \frac{1 - z_1}{1 + z_2}, \quad z_2 = \frac{1 - z_1}{1 + z_2} \quad (85)$$

which implies $z_1 = z_2 = \sqrt{2} - 1$. It can be verified that this theory has $c = 3/4$.

VII. QUASI-PERIODIC KERNELS AND THE RIEMANN HYPOTHESIS

In order get into the critical strip of $\zeta(\nu)$ with $0 < \text{Re}(\nu) < 1$, inspection of our work so far suggests that one could try to analytically continue in the spacial dimension d to complex values. However it would remain unclear what complex d actually means physically. A more physical approach is to consider non-constant, quasi-periodic kernels in fixed dimension d an integer. As we will show, $d = 1$ is sufficient to cover the whole critical strip. Because of the previously discussed bosonic instabilities for $d < 2$, one must deal with a fermionic gas. In the next subsection we will simply hypothesize a certain kernel \mathbf{K} and work out its consequences. In the subsequent subsection we will show how to obtain such a kernel from a second quantized hamiltonian with a 2-body potential that is also quasi-periodic.

A. Quasi-periodic kernel in $1d$

We assume that the 2-body potential $\mathcal{V}(\mathbf{x}, \mathbf{x}')$ in eq. (25) is translationally invariant, i.e. it depends only on the difference $\mathbf{x} - \mathbf{x}'$. In momentum space this implies that the kernel also depends only on the difference:

$$\mathbf{K}(\mathbf{k}, \mathbf{k}') = \mathbf{K}(\mathbf{k} - \mathbf{k}') \quad (86)$$

We also assume rotational invariance so that \mathbf{K} depends only on $k = |\mathbf{k}|$. Let us suppose there exists a hamiltonian which leads to the following kernel:

$$\mathbf{K}(k) = -\text{Re}(\gamma_\nu k^{2\nu-1}) \quad (87)$$

where ν is assumed to be a complex number and γ_ν is a constant. Note that the lowest order constant scattering length approximation vanishes as long as $\text{Re}(\nu) > 1/2$.

The saddle point equation, with zero chemical potential, reads:

$$\varepsilon(\mathbf{k}) = \omega_{\mathbf{k}} + \int \frac{d\mathbf{k}'}{2\pi} \text{Re}(\gamma_\nu |\mathbf{k} - \mathbf{k}'|^{2\nu-1}) \frac{1}{e^{\beta\varepsilon(\mathbf{k}')} + 1} \quad (88)$$

At low temperature, $\varepsilon(k)$ can be approximated near $k = 0$, where it continues to take the form eq. (37), with $\mu = 0$. The constant δ must satisfy the equation:

$$\delta = -Re \left[T^{\nu-1} h_\nu \text{Li}_\nu(-z_\delta) \right] \quad (89)$$

where $h_\nu = \gamma_\nu \Gamma(\nu)/2\pi$ and as before $z_\delta = e^{-\delta}$. The density and free energy however have the same expressions as in section IV specialized to $d = 1$:

$$\begin{aligned} n &= -(T/4\pi)^{1/2} \text{Li}_{1/2}(-z_\delta) \\ \mathcal{F} &= (T^3/4\pi)^{1/2} \left(\text{Li}_{3/2}(-z_\delta) + \frac{\delta}{2} \text{Li}_{1/2}(-z_\delta) \right) \end{aligned} \quad (90)$$

There are some trivial $\delta = 0$ solutions to eq. (89) which arise when the prefactor $1 - 2^{1-\nu}$ in eq. (59) equals zero. We can remove them by choosing $\gamma_\nu = ((1 - 2^{1-\nu})\Gamma(\nu))^{-1}$. We have divided by $\Gamma(\nu)$ in order to remove another trivial zero which can arise from $\Gamma(i\infty) = 0$. The coupling constant then becomes:

$$h_\nu = -\frac{2^{(\nu-3)/2}}{2\pi \sinh\left(\frac{(1-\nu)\log(2)}{2}\right)} \quad (91)$$

We can now give a clear meaning to a zero ν of the zeta function $\zeta(\nu) = 0$. Because of the relation (59), when ν is a zero, then a solution to eq. (89) is $\delta = 0$. This solution exists for *any temperature* T . By plotting the left and right hand sides of eqn. (89), one sees that in general there is another solution at $\delta \neq 0$, but the latter depends on T , and in fact for some T this solution disappears. Thus for the generic $\delta = 0$ solution, this means there are no corrections to the free energy at arbitrary temperature T , i.e. the density and free energy are the same as in a free (non-interacting) theory:

$$\begin{aligned} n &= (T/4\pi)^{1/2} (1 - \sqrt{2})\zeta(1/2) \\ \mathcal{F} &= -(T^3/4\pi)^{1/2} (1 - 1/\sqrt{2})\zeta(3/2) \end{aligned} \quad (92)$$

Note that the density is still positive since $\zeta(1/2)$ is negative: $(1 - \sqrt{2})\zeta(1/2) \approx .60649$. The pressure is also positive: $(1 - 1/\sqrt{2})\zeta(3/2) \approx .76515$.

The Riemann hypothesis would then follow from the line of reasoning: If (i) The leading contributions to the pressure of the gas are the ones calculated in this work, (ii) There is a hamiltonian that leads to the ν -dependent quasi-periodic kernel (87), and (iii) Non-zero interactions necessarily modify the pressure of the gas over a range of temperatures, then $\zeta(\nu)$ can have no zeros. In the next subsection we address (ii).

B. Real space potentials

We now show that there are indeed real space hamiltonians in $1d$ that lead to the above quasi-periodic kernel. Let $v(\mathbf{k})$ be the Fourier transform of the 2-body potential:

$$\mathcal{V}(\mathbf{x} - \mathbf{x}') = \int \frac{d^d \mathbf{k}}{(2\pi)^d} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')} v(\mathbf{k}) \quad (93)$$

Using eq. (30) one finds

$$\mathbf{K}(\mathbf{k}) = -\frac{1}{4} (v(\mathbf{k}) + v(-\mathbf{k}) + 2s v(0)) \quad (94)$$

We have included the statistics parameter s in order to point out some features.

Consider the following potential:

$$\mathcal{V}(\mathbf{x}) = Re \left(\frac{b_\nu}{|\mathbf{x}|^{2\nu}} \right) \quad (95)$$

where b_ν is a constant. If b_ν is real and $\nu = \sigma/2 + i\alpha$ then the potential is quasi-periodic in $\log |x|$:

$$\mathcal{V}(\mathbf{x}) = \frac{b_\nu \cos(2\alpha \log |\mathbf{x}|)}{|\mathbf{x}|^\sigma} \quad (96)$$

This model is not known to be integrable, unlike the case of a delta-function potential, thus there is no known exact thermodynamic Bethe ansatz. The lowest order corrections are the ones calculated in the last section.

When the particles are fermionic with $s = -1$ the $v(0)$ term in eq. (94) precisely renders the kernel well behaved as $k \rightarrow 0$. The kernel would be singular if the particles were bosonic. The kernel is then given by the following integral:

$$\mathbf{K}(k) = Re \left(2b_\nu \int_0^\infty dx x^{-2\nu} \sin^2(kx/2) \right) \quad (97)$$

This integral is convergent if the following condition is satisfied:

$$1/2 < Re(\nu) < 3/2 \quad (98)$$

If the above condition is met, the kernel is the following:

$$\mathbf{K}(k) = -Re [b_\nu k^{2\nu-1} \sin(\pi\nu) \Gamma(1 - 2\nu)] \quad (99)$$

Finally we can chose b_ν in order to obtain the kernel in eq. (91). Using the identity

$$\sin(\pi\nu) \Gamma(1 - 2\nu) \Gamma(\nu) = \sqrt{\pi} 2^{-2\nu} \Gamma(1/2 - \nu) \quad (100)$$

this fixes b_ν to be:

$$b_\nu = -\frac{2^{(5\nu-3)/2}}{\sqrt{\pi}\Gamma\left(\frac{1-2\nu}{2}\right)\sinh\left(\frac{(1-\nu)\log(2)}{2}\right)} \quad (101)$$

The condition (98) is precisely what one needs for the RH. What is then the meaning of the known zeros at $Re(\nu) = 1/2$? These are models with the kernel (87) that give vanishing leading corrections to the pressure. The calculations of this section show that such a kernel does not arise in a convergent manner from a real space potential since the condition (98) is violated. Note that the Γ function in eq. (99) develops a pole at $\nu = 1/2$, which suggests that an additional low-energy regularization could still lead to sensible models with the kernel (87) that provide physical realizations of the Riemann zeros.

VIII. CONCLUSIONS

We have shown how the formalism developed in [1] can lead to new results for the quantum statistical mechanics of interacting gases of bosons and fermions. Our main results were summarized in the introduction.

Clearly our most interesting result is the formulation of the Riemann hypothesis in the present context. We have essentially given a physical argument from which it follows. In order to develop this argument into a rigorous mathematical proof, one mainly needs to give a firm foundation to the theoretical methods used here, namely the framework developed in [1]. One also needs to understand more rigorously how the contributions considered in this paper are the leading ones, i.e. one needs clearer control of the approximations we have made.

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X. APPENDIX

In this appendix we explain how the $\nu \rightarrow 1 - \nu$ duality of Riemann's zeta function $\zeta(\nu)$ can be understood as a special modular transformation in a Lorentz-invariant theory.

Consider a free quantum field theory of massless bosonic particles in $d + 1$ spacetime dimensions with euclidean action $S = \int d^{d+1}x (\partial\phi)^2$. The geometry of euclidean spacetime is take to be $S^1 \times R^d$ where the circumference of the circle S^1 is β . We will refer to the S^1 direction as “ \hat{x} ”. Endow the flat space R^d with a large but finite volume as follows. Let us refer to one the directions perpendicular to \hat{x} as \hat{y} with length L and let the remaining $d - 1$ directions have volume A .

Let us first view the \hat{x} direction as compactified euclidean time, so that we are dealing with finite temperature $T = 1/\beta$. As a quantum statistical mechanical system, the partition function in the limit $L, A \rightarrow \infty$ is

$$Z = e^{-\beta V \mathcal{F}(\beta)} \quad (102)$$

where $V = L \cdot A$ and \mathcal{F} is the free energy density. Standard results give:

$$\mathcal{F}(\beta) = \frac{1}{\beta} \int \frac{d^d \mathbf{k}}{(2\pi)^d} \log(1 - e^{-\beta k}) \quad (103)$$

The euclidean rotational symmetry allows one to view the above system with time now along the \hat{y} direction. In $1d$, interchanging the role of \hat{x} and \hat{y} is a special case of a modular transformation of the torus. In this version, the problem is a zero temperature quantum mechanical system with a finite size β in one direction, and the total volume of the system is $V' = \beta \cdot A$. The quantum mechanical path integral leads to

$$Z = e^{-L E_0(A, \beta)} \quad (104)$$

where E_0 is the ground state energy. Let $\mathcal{E}_0 = E_0/V'$ denote the ground state energy per volume. Comparing the two “channels”, their equivalence requires $\mathcal{E}_0(\beta) = \mathcal{F}(\beta)$. In this finite-size channel, the modes of the field in the \hat{x} direction are quantized with wave-vector $k_x = 2\pi n/\beta$, and the calculation of \mathcal{E}_0 is as in the Casimir effect:

$$\mathcal{E}_0 = \frac{1}{2\beta} \sum_{n \in \mathbb{Z}} \int \frac{d^{d-1} \mathbf{k}}{(2\pi)^{d-1}} (\mathbf{k}^2 + (2\pi n/\beta)^2)^{1/2} \quad (105)$$

The free energy density \mathcal{F} can be calculated using $\int d^d \mathbf{k} = 2\pi^{d/2}/\Gamma(d/2)$. For $d > 0$ the integral is convergent and one finds

$$\mathcal{F} = -\frac{1}{\beta^{d+1}} \frac{\Gamma(d+1)\zeta(d+1)}{2^{d-1}\pi^{d/2}\Gamma(d/2)d} \quad (106)$$

For the Casimir energy, \mathcal{E}_0 involves $\sum_{n \in \mathbb{Z}} |n|^d$ which must be regularized. As is usually done, let us regularize this as $2\zeta(-d)$. Then:

$$\mathcal{E}_0 = -\frac{1}{\beta^{d+1}} \pi^{d/2} \Gamma(-d/2) \zeta(-d) \quad (107)$$

Define the function

$$\xi(\nu) \equiv \pi^{-\nu/2} \Gamma(\nu/2) \zeta(\nu) \quad (108)$$

Then the equality $\mathcal{E}_0 = \mathcal{F}$ requires the identity:

$$\xi(\nu) = \xi(1 - \nu) \quad (109)$$

The above relation is a known functional identity that can be proven using complex analysis. Thus we have demonstrated that ζ function regularization of the Casimir energy is consistent with a modular transformation to the finite-temperature channel. On the other hand, our calculations can be viewed as a proof of the identity (109) based on physical consistency.

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- [28] There are also some convergence problems with the usual TBA, which are avoided in the present work.